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NEW REPRESENTATION AND METHOD OF SOLVING FREDHOLM INTEGRAL EQUATIONS

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It is shown that the solution of the Fredholm Integral equation

$$u(t, x) = g(t) + \int_0^x k(t, y)\, u(y, x)\, dy, \quad 0 \leqslant t \leqslant x \leqslant X \tag{0.1}$$

can be expressed in terms of the two functions $\Phi = \Phi(t, x)$ and $\Psi = \Psi(y, x)$, each of which depends only on two arguments. There is obtained a complete Cauchy system for these two functions and for a third auxiliary function $Z = Z(x)$.

More than twenty years ago, Krein [1] derived a partial differential equation for the Fredholm resolvent. Several years later this equation was obtained independently by Bellman [2] on the basis of a variational principle.

The purpose of the present paper is to deduce a complete system of differential equations, one of which is the Krein — Bellman equation, and then to obtain a representation of the solution of the Fredholm integral equation by using functions $\Phi = \Phi(t, x)$ and $\Psi = \Psi(y, x)$ in place of resolvents containing three arguments. Reduction of Fredholm integral equations to a Cauchy system is described in detail in the book [3]. The ideas of this paper are an extension of the Sobolov's [4] ideas.

1. Introducing the resolvent K, the solution of (0.1) can be represented in the form

$$u(t, x) = g(t) + \int_0^x K(t, y, x)\, g(y)\, dy, \quad 0 \leqslant t \leqslant x \tag{1.1}$$

The resolvent K satisfies the integral equation

$$K(w, y, x) = k(w, y) + \int_0^x K(w, \zeta, x)\, k(\zeta, y)\, d\zeta, \quad 0 \leqslant w, y \leqslant x \leqslant X \tag{1.2}$$

Let us introduce the auxiliary function Φ as a solution of the integral equation

$$\Phi(t, x) = k(t, x) + \int_0^x k(t, y)\, \Phi(y, x)\, dy, \quad 0 \leqslant t \leqslant x \leqslant X \tag{1.3}$$

Comparing the integral equation (1.3) with the integral equation for u_x obtained by differentiating (0.1) with respect to x and using their linearity, we have

$$u_x(t, x) = \Phi(t, x)\, u(x, x) \tag{1.4}$$

Let us define a new function Ψ as

$$\Psi(y, x) = K(x, y, x), \quad 0 \leqslant y \leqslant x \leqslant X \tag{1.5}$$

Then (1. 1) can be written as follows for $t = x$:

$$U(x) = g(x) + \int_0^x \Psi(y, x) g(y) dy \tag{1.6}$$

$$U(x) = u(x, x) \tag{1.7}$$

Integrating both sides of (1. 4) with respect to x and using (1. 7), we obtain

$$u(t, x) = U(t) + \int_t^x \Phi(t, w) U(w) dw \tag{1.8}$$

Substituting (1. 6) into (1. 8), the solution of the Fredholm equation can be written as follows [5]:

$$u(t, x) = g(t) + \int_0^t \Psi(y, t) g(y) dy + \tag{1.9}$$

$$\int_t^x \Phi(t, w) \left[g(w) + \int_0^w \Psi(y, w) g(y) dy \right] dw$$

This formula yields the solution $u(t, x)$ expressed in terms of the auxiliary functions Φ and Ψ which depend on two arguments (the resolvent K depends on three arguments).

It is seen from (1. 5) and (1. 2) that Ψ satisfies the integral equation

$$\Psi(y, x) = k(x, y) + \int_0^x \Psi(y', x) k(y', y) dy', \quad 0 \leqslant y \leqslant x \leqslant X \tag{1.10}$$

2. If the substitution

$$g(t) = k(t, y), \quad 0 \leqslant t \leqslant x \tag{2.1}$$

is made in (1. 9), then

$$u(t, x) = K(t, y, x) \tag{2.2}$$

Substituting the relation (2. 1) into (1. 6) and comparing the equation obtained with (1. 10), we have $U(t) = \Psi(y, t)$. Using (2. 2) and (1. 8), we obtain

$$K(t, y, x) = \Psi(y, t) + \int_t^x \Phi(t, w) \Psi(y, w) dw \tag{2.3}$$

Recalling the integral equations (1. 3) and (1. 10) for Φ and Ψ, we see that Φ and Ψ are defined for $x < t$ and $y < t$, hence, (2. 3) is suitable for all values of t, y. Differentiating (2. 3) with respect to x, we obtain partial differential equations for K depending on the functions Φ and Ψ

$$\begin{aligned} &K_x(t, y, x) = \Phi(t, x)\Psi(y, x) \\ &K(t, y, y) = \Phi(t, y), \; y > t \\ &K(t, y, t) = \Psi(y, t), \; t > y \end{aligned} \tag{2.4}$$

Equations (1.2) and (1.3) yield

$$\Phi(t, x) = K(t, x, x), \; 0 \leqslant t \leqslant x \tag{2.5}$$

Equations (2.4) and (1.3) yields

$$K_x(t, y, x) = K(t, x, x)\, K(x, y, x), \; 0 \leqslant t, \; y \leqslant x \tag{2.6}$$

i.e., we arrive at the Krein — Bellman equation.

3. Let us derive differential equations for the functions Φ, Ψ, K and Z.

We start with the function $\Phi(t, x)$. We differentiate (1.3) with respect to x, which yields

$$\Phi_x(t, x) = A(t, x) + \int_0^x k(t, y)\, \Phi_x(y, x)\, dy \tag{3.1}$$

$$A(t, x) = k_x(t, x) + k(t, x)\Phi(x, x) \tag{3.2}$$

Let us note that (3.1) for Φ_x agrees with (0.1) for $u(t, x)$ if we substitute

$$g(t) = A(t, x) \tag{3.3}$$

in (0.1).

The solution for u is given by (1.8) and (1.6), hence the solution for Φ can also be expressed by the relations (1.8) and (1.6) upon compliance with condition (3.3). Similarly to (1.8), we obtain

$$\Phi_x(t, x) = R(t, x) + \int_t^x \Phi(t, w)\, R(w, x)\, dw, \quad 0 \leqslant t \leqslant x \leqslant X \tag{3.4}$$

$$R(t, x) = A(t, x) + \int_0^x \Psi(y, x)\, A(y, x)\, dy$$

Let us derive a differential equation for the function $\Psi(y, x)$. To do this, we differentiate both sides of (1.10) with respect to x

$$\Psi_x(y, x) = B(y, x) + \int_0^x \Psi_x(w, x)\, k(w, x)\, dw \tag{3.5}$$

$$B(y, x) = k_x(x, y) + \Psi(x, x)k(x, y) \tag{3.6}$$

We now examine the auxiliary integral equation

$$v(y, x) = h(y) + \int_0^x v(w, x)\, k(w, y)\, dw, \quad 0 \leqslant y \leqslant x \leqslant X \tag{3.7}$$

Differentiating (3.7) with respect to x and comparing the result with (1.10), we obtain

$$v_x(y, x) = \Psi(y, x)v(x, x) \tag{3.8}$$

Integrating (3.8) with respect to x, we obtain

$$v(y, x) = v(y, y) + \int_y^x \Psi(y, w)\, v(w, w)\, dw \tag{3.9}$$

Let us introduce the resolvent of the kernel $L(w, y, x)$, then the solution of the equation is expressed as follows:

$$v(y, x) = h(y) + \int_0^x h(w)\, L(w, y, x)\, dw, \quad 0 \leqslant w \leqslant x \tag{3.10}$$

The solution of (3.9) will be

$$v(y, x) = V(y) + \int_y^x \Psi(y, w)\, V(w)\, dw \tag{3.11}$$

$$V(x) \equiv v(x, x) = h(x) + \int_0^x h(w)\, L(w, x, x)\, dw \tag{3.12}$$

We determine the function $L(w, x, x)$. To do this, we equate the right sides of (3.7) and (3.10) and obtain

$$\int_0^x v(w, x)\, k(w, y)\, dw = \int_0^x h(w)\, L(w, y, x)\, dw \tag{3.13}$$

In conformity with (3.10), we write an expression for $v(w, x)$, where we replace the variable of integration by ζ. Substituting this expression for $v(w, x)$ under the integral sign in (3.13), we obtain

$$\int_0^x h(w)\, k(w, y)\, dw + \int_0^x k(w, y)\, dw \int_0^x h(\zeta)\, L(\zeta, w, x)\, d\zeta = \int_0^x h(w)\, L(w, y, x)\, dw \tag{3.14}$$

Let us replace w by ζ and ζ by w in the second term in the left side of (3.14) and let us collect terms in $h(w)$. Taking into account that $h(w)$ is an arbitrary function, we hence obtain the integral equation

$$L(w, y, x) = k(w, y) + \int_0^x L(w, \zeta, x)\, k(\zeta, y)\, d\zeta \tag{3.15}$$

Taking into account the Fredholm integral equation (3.7) and its solution (3.10), we write the solution for the Fredholm integral equation (3.15) in the form

$$L(w, y, x) = k(w, y) + \int_0^x k(w, \zeta)\, L(\zeta, y, x)\, d\zeta \tag{3.16}$$

Substituting $y = x$ into (3.16), and $t = w$ into (1.3) and equating the results, we obtain

$$L(w, x, x) = \Phi(w, x), \quad 0 \leqslant w \leqslant x \leqslant X \tag{3.17}$$

Using (3. 17), we write the solution for $v(y, x)$ determined by (3. 11) and (3. 12) in the form

$$v(y, x) = V(y) + \int_y^x \Psi(y, w) V(w)\, dw \tag{3.18}$$

$$V(x) = h(x) + \int_0^x h(w)\, \Phi(w, x)\, dw \tag{3.19}$$

Comparing (3. 5) and (3. 7), we remark that Ψ_x satisfies the integral equation (3. 7) if we set

$$h(y) = B(y, x) \tag{3.20}$$

We hence conclude that under the condition (3. 20) the solution for Ψ_x agrees with the solution for $v(y, x)$ determined by (3. 18) and (3. 19). It is hence possible to write the solution for Ψ_x in the form

$$\Psi_x(y, x) = P(y, x) + \int_y^x \Psi(y, w)\, P(w, x)\, dw, \quad 0 \leqslant y \leqslant x \tag{3.21}$$

$$P(y, x) = B(y, x) + \int_0^x B(w, x)\, \Phi(w, x)\, dw$$

Let us introduce the new function

$$Z(x) = \Phi(x, x), \; 0 \leqslant x \leqslant X \tag{3.22}$$

Differentiating (3. 22) with respect to x we obtain

$$\frac{dZ(x)}{dx} = \Phi_1(x, x) + \Phi_2(x, x) \tag{3.23}$$

$$\Phi_1(x, x) = \left[\frac{\partial \Phi(y, x)}{\partial y}\right]_{y=x}, \quad \Phi_2(x, x) = \left[\frac{\partial \Phi(y, x)}{\partial x}\right]_{y=x}$$

Differentiating (1. 3) for Φ with respect to t and setting $t = x$, we obtain

$$\Phi_1(x, x) = k_1(x, x) + \int_0^x k_1(x, \zeta)\, \Phi(\zeta, x)\, d\zeta, \quad k_1(x, x) = \left[\frac{\partial k(t, y)}{\partial t}\right]_{t=y=x} \tag{3.24}$$

The function $\Phi_2(x, x)$ agrees with $\Phi_x(t, x)$. If we substitute $t = x$, we obtain in conformity with the first equation in (3. 4)

$$\Phi_2(x, x) = R(x, x) \tag{3.25}$$

Substituting (3. 24) and (3. 25) into (3. 23), we obtain an integro-differential equation for $Z(x)$

$$\frac{dZ(x)}{dx} = k_1(x, x) + \int_0^x k_1(x, \zeta)\, \Phi(\zeta, x)\, d\zeta + R(x, x) \tag{3.26}$$

Here, according to the second relationship in (3.4) and (3.2)

$$R(x,x)=A(x,x)+\int_0^x \Psi(\zeta,x)A(\zeta,x)\,d\zeta \tag{3.27}$$

$$A(x,x)=k_2(x,x)+k(x,x)\Phi(x,x),\quad k_2(x,x)=\left[\frac{\partial k(t,y)}{\partial y}\right]_{t=y=x}$$

4. Let us determine the initial conditions for the functions Φ, Ψ and Z, K. We obtain the initial conditions for Φ from (3.22) by substituting $x=t$

$$\Phi(t,t)=Z(t) \tag{4.1}$$

In order to obtain the initial conditions for K, let us recall (2.5) and (1.5). Substituting $t=y=x$ therein, and using (4.1), we obtain

$$\Phi(x,x)=\Psi(x,x)=K(x,x,x)=Z(x) \tag{4.2}$$

It is seen from (4.2) that the initial condition for Ψ is

$$\Psi(y,y)=Z(y) \tag{4.3}$$

The initial condition for $Z(x)$ is obtained by substituting $x=0$ in (1.2) and (4.2)

$$Z(0)=k(0,0) \tag{4.4}$$

The initial conditions for the resolvent K are given by the last two relationships in (2.4). S.L. Sobolev [6] examined this question.

5. Thus, a system of integro-differential equations (3.4), (3.21), (3.26) and (2.4) (the first equation) with initial conditions (4.1), (4.3), (4.4) and (2.4) (the last two relationships) has been obtained. The integrals can be approximated by a finite sum and a computational technique can be used. After the functions Φ and Ψ have been determined from (3.4) and (3.21), the function u can be determined by (1.8) and (1.6).

The differential equations for Φ, Ψ and Z are a complete system which can be used for the numerical determination of these three functions. If the Krein — Bellman equation (2.6) and the initial conditions from (2.4) are appended, then a complete system of equations to determine Z, Φ, Ψ and K is obtained.

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